Exercise Problems:

1) If R is the region bounded by planes $x=0$, $y=0$ $z=0$ and $x+y+z=1$, show that $\iiint_R z\,dx\,dy\,dz = \frac{1}{24}$

Solution: The region of integration R is expressed as $0 \le z \le 1-x-y$, $0 \le y \le 1-x$, $0 \le x \le 1$

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} z\,dz\,dy\,dx = \int_0^1\int_0^{1-x} \frac{z^2}{2}\Big|_0^{1-x-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{1-x} [1+x^2+y^2-2x+2xy-2y]\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 \left[y+x^2y+\frac{y^3}{3}-2xy+2x\frac{y^2}{2}-\frac{2y^2}{2}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[y+x^2y+\frac{y^3}{3}-2xy+xy^2-y^2\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \Big[\underline{(1-x)} + \underline{x^2(1-x)} + \frac{(1-x)^3}{3} - \underline{2x(1-x)} + \underbrace{x(1-x)^2} - \underbrace{(1-x)^2}\Big]\,dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x)(1+x^2-2x) + (1-x)^2(x-1) + \frac{(1-x)^3}{3}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x)(1-x)^2 + (1-x)^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \frac{1}{2}\int_0^1 \left[\cancel{(1-x)^3} - \cancel{(1-x)^3} + \frac{(1-x)^3}{3}\right] dx$$

$$= \frac{1}{2}\int_0^1 \frac{(1-x)^3}{3}\,dx$$

$$= \frac{1}{2}\left[-\frac{(1-x)^4}{4\times 3}\right]_0^1$$

$$= \frac{1}{24}\left[-\cancel{(1-1)^4}^{\,0} + (1-0)\right]$$

$$= \frac{1}{24}(1) = 1/24$$

H.W

2) If R is the positive octant of the sphere $x^2+y^2+z^2=a^2$ show that $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz = \frac{\pi}{10}a^5$.

3) If R is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$, show that

$$\iiint_R (x+y+z)\,dx\,dy\,dz = 1/8$$

4) If R is the region bounded by $0<x<10$, $0<y<1$, $0<z<1$, show that

$$\iiint_R xy^2z^3\,dx\,dy\,dz = \frac{1}{24}$$

5) If R is the region bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=a$ $(a>0)$, Show that

$$\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz = \frac{a^5}{20}$$